ASSOCIAZIONE NAZIONALE ITALIANA DI INGEGNERIA SISMICA ANIDIS
Italian National Association of Earthquake Engineering

XVIII CONVEGNO ANIDIS
L'ingegneria sismica in Italia
ASCOLI PICENO 2019
15-19 Settembre

Evaluation and reduction of the truncation effects on modal flexibility-based damage-sensitive features in frame buildings with mass irregularities

Giacomo Bernagozzi, Luca Landi, Pier Paolo Diotallevi
Dipartimento di Ingegneria Civile, Chimica, Ambientale e dei Materiali (DICAM), Università di Bologna, Viale Risorgimento 2, 40136 Bologna, Italy

Keywords: vibration-based damage detection; modal flexibility-based interstory drift; flexibility truncation error; mass irregular structure; mass proportional load; shear-type frame building

ABSTRACT
Recent approaches for vibration-based damage detection in building structures are based on the calculation of modal flexibility (MF) based deflections. These deflections are estimated by applying uniform loads (UL) to experimentally derived modal flexibility matrices of the structures, and the interstory drifts are considered as damage-sensitive features (DSFs). However, in practical applications only a limited number of modes is usually identified from vibration data, and thus modal truncation effects are introduced on the deflections. To address this problem a mass proportional load (MPL) has been proposed, in a previous study by the authors, as an alternative to the uniform load with the aim of reducing the truncation effects on the displacement components of MF-based deflections of structures with mass irregularities. The objective of this work is to investigate further the mentioned problem by evaluating the truncation errors that affect the MF-based interstory drifts of frame buildings (i.e. the DSFs that can be conveniently used for damage detection). For structures with mass irregularities the truncation errors related to the application of the proposed MPL are thus compared with those related to the UL. Parametric studies were carried out on numerical models of shear-type frame buildings by considering structures with a different number of stories and various distributions of the story mass ratios, and by considering different subsets of modes included in the calculations.

1 INTRODUCTION

Assessing the condition and the health of civil structures over time and especially after potential damaging events, such as earthquakes, is still a complex and challenging task. To this end, promising strategies and techniques are the ones that belong to the field of vibration-based Structural Health Monitoring (SHM). Civil structures, such as building or bridge structures, can be conveniently instrumented with vibration sensors, and tested during their normal operating conditions under ambient excitations. Data acquired over time or, in general, at different time instants can then be analyzed, and one of the main goals that can be attained is to detect the eventual presence of damage in the structure (Sohn et al. 2003), (Farrar and Worden 2013).

In vibration-based damage detection and localization of structures, simple but effective techniques are the so-called modal flexibility-based techniques. According to these techniques the modal parameters of the structures are identified from the vibration data and then used to assemble an experimentally-derived model. Such model consists, specifically, in the modal flexibility matrix of the structure. In the framework of such techniques, as shown in the works of (Zhang and Aktan 1998), (Koo et al. 2010), the more advanced and refined approaches are based on an additional important main step – i.e. structural deflections are estimated from the modal flexibility-based models by applying specific loads which are denoted as inspection loads. Such loads in the majority of the approaches are uniform loads (UL) which are applied at all the DOFs of the identified structural model. Different variants of the deflection-based methods have been developed, depending on whether such methods are, for example, applied to bridges (Koo et al. 2008) or buildings (Koo et al. 2010), (Koo et al. 2011), (Bernagozzi et al. 2017b), (Bernagozzi et al. 2018). Referring to the specific case of building structures, the interstory drifts computed from the modal flexibility-based deflections due to positive shear inspection loads (PSIL) are assumed as the damage sensitive features (DSFs).

An important circumstance has to be considered when dealing with methods based on modal flexibility and related deflections – i.e. not all the structural modes can be identified in modal testing and identification of civil structures. This is especially true for civil structures tested under ambient vibrations. The input in such case has in general a wide frequency content, however, since the test is executed using exclusively the natural excitations, some structural modes may not be excited, and thus they may not be identified (Brincker and Ventura 2015). This implies that the modal flexibility is usually assembled using a limited number of modes, and thus is inevitably affected by intrinsic errors - i.e. discrepancies with respect to the theorical flexibility assembled using all the modes. These errors are known as flexibility truncation errors (Zhang and Aktan 1998), and they in turn also affect the modal flexibility-based deflections.

According to (Zhang and Aktan 1998), the objective of a modal truncation error analysis is to determine how many modes need to be included in order to obtain adequate estimates of the modal flexibility matrices and the modal flexibility-based deflections derived from an experimental test. These analyses can be performed, for example, using numerical models, and they provide useful information which can be adopted to design and execute the testing phase. By performing numerical analyses on a 10 DOF discrete system of masses and springs and on a three-span bridge model, (Zhang and Aktan 1998) showed that the modal flexibility is more sensitive to the number of included modes than the uniform load deflection. This last is, on the contrary, more accurately estimated, as long as the first main modes are included. In (Zhang and Aktan 1998) the truncation error study has been performed by comparing truncated and nontruncated solutions. In addition, another criterion for truncation error analysis is mentioned, but not applied, in the work by (Zhang and Aktan 1998) – i.e. evaluating the mass participation factors for the considered structural modes. In (Bernagozzi et al. 2017a) an approach has been proposed to predict the modal truncation effects on the displacement components of modal flexibility-based deflections due to generic loads. The approach proposed in the mentioned work is based on the evaluation for the considered modes of a proposed index termed Load Participation Factor (LPF), and it can be considered as a generalization of the approach by (Zhang and Aktan 1998), based on mass participation factor, to the case of generic loads and to the case of structures with generic distributions of the masses. The proposed approach was validated through numerical simulations on a frame building and using the experimental data of a steel frame structure tested under ambient vibrations. In (Bernagozzi et al. 2017a) it was also shown that the truncation errors on the displacement components of uniform load deflections are in general non negligible especially for structures with mass irregularities. To reduce such truncation errors, in the mentioned work a special inspection load, termed Mass Proportional Load (MPL), has been proposed as an alternative to the commonly used uniform load. Through numerical analyses on models of frame buildings and simply supported beams with mass irregularities, it was shown that in general the truncation errors on the displacement components of the MPL deflections are lower than the corresponding errors on the UL deflections.

Referring specifically to the problem of reducing (not predicting) the modal truncation effects, in (Bernagozzi et al. 2017a) all the performed analyses have been focused on reducing the errors on the displacement components of the modal flexibility-based deflections. For building structures, however, the interstory drifts calculated from the deflections are also important parameters. As already mentioned, in fact, such parameters are considered as damage sensitive features, for example, in the output-only vibration-based damage detection method proposed by (Koo et al. 2010), which, for the sake of convenience, is simply denoted in the present paper as PSIL method. Thus, by considering that the evaluation of the modal flexibility-based interstory drifts from an experimental vibration test on a real building structure is in general performed using a limited number of structural modes, it is evident that reducing the modal truncation effects on such parameters is a desirable result.

The first objective of this paper is to evaluate the truncation effects that affect the interstory drifts computed from modal flexibility-based deflections due to a uniform load of building structures with irregular in elevation mass distributions. Then, the paper also aims to evaluate if such truncation errors can be reduced by applying loads different from the uniform load, such as, for example, a mass proportional load – i.e. the same load that was adopted in the previous work by the authors presented in (Bernagozzi et al. 2017a), where, on the contrary, all the analyses have been focused on reducing the errors on the displacement components of the deflections. The present paper can thus be considered as a continuation of the mentioned previous work by the authors. In the present paper, numerical analyses and parametric studies related to the study of the flexibility truncation errors were performed on various models of shear-type buildings with different distributions of the structural masses and a different number of stories.

2 ESTIMATION FROM VIBRATION DATA OF MODAL FLEXIBILITY MATRICES, STRUCTURAL DEFLECTIONS AND INTERSTORY DRIFTS RELATED TO FRAME BUILDINGS

The type of structures that are considered in the paper are regular and plan-symmetric frame buildings which can be modelled as planar shear-type structures. Such structures are the same structures for which the PSIL method for damage detection was developed (Koo et al. 2010). Let us assume that the building structure is subjected to an ambient vibration (AV) test, and that the objective is to estimate the modal flexibility matrix of the structure – i.e. an experimentally-derived mechanical model of the structure that can be used, for example, for damage detection purposes. When conducting this operation, to have an accurate and reliable estimate of the modal flexibility it is important to acquire a sufficient number of vibration measurements in different spatial locations. For example, as done in (Koo et al. 2010), it is assumed that the acceleration vibration measurements are available at all the stories of the building. This result can be obtained if the number of sensors to be used in the AV test is equal to the number of the stories. Alternatively, if the number of sensors is lower than the number of the stories, the vibration data can be acquired in multiple data sets by adopting both reference and roving sensors (i.e. fixed and moving sensors) in different experimental test setups (Brincker and Ventura 2015).

Starting from the recorded vibration data the modal parameters can be extracted using any output-only modal identification or operational modal analysis technique (Brincker and Ventura 2015). Then, the modal flexibility matrix $\boldsymbol{F_r}_{\,n\times n}$ of the structure can be assembled as follows

$$\boldsymbol{F_r} = \boldsymbol{\Phi_r}\,\boldsymbol{\Lambda_r}^{-1}\,\boldsymbol{\Phi_r}^{\mathrm{T}} \tag{1}$$

where r is the number of the identified modes with $r \leq n$, n is the number of the DOFs (equal to the number of the stories for a planar shear-type building), $\boldsymbol{\Phi_r}_{\,n\times r}$ is a modal matrix formed by r columns and each column contains a mass-orthogonal and mass-normalized real mode shape vector, $\boldsymbol{\Lambda_r}_{\,r\times r}$ is a diagonal spectral matrix which contains the square of the first r natural circular frequencies ω_i^2 on the main diagonal, and i=1…r is the mode index. If $r < n$, the modal flexibility matrix of Equation 1 is affected by modal truncation errors; if $r = n$ no truncation errors are present. Of course, in any case, uncertainties on the components of the modal flexibility matrix are always present since such quantities are estimated from real vibration data inevitably affected by noise. To estimate the modal flexibility, as shown in Equation 1, mass normalized mode shapes are required. These scaled mode shapes can be obtained using a modal scaling approach, such as for example the mass change method (Bernal 2004), (Aenlle et al. 2010), or by simply considering in the calculations an a-priori estimate of the system mass matrix, as done in (Koo et al. 2010).

Starting from the modal flexibility matrix, the modal flexibility-based deflection $\boldsymbol{x_r}_{\,n\times 1}$ due to a generic load $\boldsymbol{p}_{n\times 1}$ can be determined as follows

$$\boldsymbol{x_r} = \boldsymbol{F_r}\,\boldsymbol{p} \tag{2}$$

According to the damage detection method applicable to shear-type buildings presented by (Koo et al. 2010), the considered inspection loads should be Positive Shear Inspection Loads (PSIL), i.e. loads that generate positive shear forces in all the stories of the building. However, among the loads that have this characteristic, in (Koo et al. 2010) it is suggested to consider a uniform load (UL) with components that are equal to one – i.e. an n×1 vector equal to $[1 \;\; 1 \;\; \dots \;\; 1]^{\mathrm{T}}$, which is the same load that was considered also in the work by (Zhang and Aktan 1998). In the method presented by (Koo et al. 2010), this load is applied to modal flexibility matrices related to both the undamaged and the potentially damaged states.

For a shear-type building the interstory drifts can be computed from the MF-based deflection as follows

$$d_{r,j} = \begin{cases} x_{r,j} - x_{r,j-1} & \text{for } j=2\dots n \\ x_{r,j} & \text{for } j=1 \end{cases} \tag{3}$$

where $d_{r,j}$ is the drift at the j-th story and $x_{r,j}$ is the displacement component related to the j-th DOF of the structure, both of them estimated by considering in the calculations a number of modes equal to r. The parameters obtained from Equation 3 – i.e. modal flexibility-based interstory drifts – are considered as damage-sensitive features and used for damage detection, localization and also quantification according, for example, to the criteria of the methods proposed in (Koo et al. 2010), (Koo et al. 2011).

3 TRUNCATION ERROR ANALYSIS ON MF-BASED DEFLECTIONS AND INTERSTORY DRIFTS OF FRAME BUILDINGS

3.1 *Comparison between the truncated and non-truncated solutions*

The analysis of the flexibility truncation errors can be performed, according to (Zhang and Aktan 1998), using a preliminary numerical model of the structure. Among the different approaches that are mentioned in the work by (Zhang and Aktan 1998), the more simple and intuitive one is to directly compare the truncated and non-truncated solutions, and such solutions can be, for example, components of the modal flexibility matrix or displacement components of the MF-based deflections. The truncated solutions are obtained by considering a limited number of structural modes, while the non-truncated solutions contain all the structural modes and are equivalent to the solutions that can be obtained using the exact static flexibility matrix of the structure.

The relative modal truncation error on the j-th displacement component of the modal flexibility-based deflection can be evaluated as follows

$$\varepsilon_{x,r}^{j} = \frac{x_{r,j} - x_{n,j}}{x_{n,j}} \tag{4}$$

where $x_{r,j}$ is the truncated displacement computed from the modal flexibility matrix $\boldsymbol{F_r}$ assembled using r modes and $x_{n,j}$ is the exact displacement computed from the static flexibility matrix $\boldsymbol{F_n}$ (which is equivalent to the modal flexibility matrix assembled using all modes). The criterion was presented in the work by (Zhang and Aktan 1998) for the case of uniform load deflections, but, as shown in (Bernagozzi et al. 2017a), the criterion can also be applied to deflections due to generic loads.

For the purposes of the present paper, the criterion expressed in Equation 4 is applied to the modal flexibility-based interstory drifts of building structures. Thus, the relative modal truncation error on the interstory drift related to the j-th story and due to a generic load can be evaluated as

$$\varepsilon_{d,r}^{j} = \frac{d_{r,j} - d_{n,j}}{d_{n,j}} \tag{5}$$

where, similarly to Equation 4, the terms $d_{r,j}$ and $d_{n,j}$ represent the truncated and non-truncated solutions, respectively.

Then, it can be also of interest and convenient to quantify the amount of the truncation effects that affect all the interstory drifts of the building using a unique single parameter. In the analyses of the present paper, the root-mean-square (RMS) criterion was applied as follows

$$\varepsilon_{d,r}^{RMS} = \sqrt{\frac{1}{n} \sum_{j=1}^{n} \varepsilon_{d,r}^{j\ 2}} \tag{6}$$

where $\varepsilon^{j}_{d,r}$ is the relative truncation error on the interstory drift related to the j-th story evaluated using Equation 5. The criterion presented in Equation 6 was adapted from the one presented in (Bernagozzi et al. 2017a), where, on the contrary, the root-mean-square error was calculated for the whole modal flexibility-based deflection starting from the truncation errors on the displacement components of all the DOFs.

3.2 *Truncation error analysis using the mass and load participation factors*

Alternative approaches for truncation error analysis do not imply a comparison between the truncated and non-truncated solutions. On the contrary, such approaches are based on the evaluation of specific participation factors for the modes that are included in the modal flexibility matrices and related deflections. Thus, for these approaches for truncation error analysis having the knowledge of all the modes of the structure is, in principle, not required.

According to (Zhang and Aktan 1998), the analysis of the truncation effects introduced on modal flexibility matrices and uniform load deflections (assembled using a limited number of modes equal to r with r < n) can be performed by evaluating, for the considered r modes, the cumulative mass participation factor (MPF) of the structure

$$\mu_r = \sum_{i=1}^{r} \mu_{(i)} \tag{7}$$

where $\mu_{(i)}$ is the mass participation factor related to the i-th mode of the structure. This last quantity can be evaluated as follows

$$\mu_{(i)} = \frac{\Gamma_i^2}{\boldsymbol{\Gamma}^{\mathrm{T}} \boldsymbol{\Gamma}} \tag{8}$$

if a structure with a diagonal mass matrix (i.e. the case for a planar shear-type building) and mass-normalized mode shapes are considered. In Equation 8 the term Γ_i is the modal participation factor related to the i-th mode of the structure, which can be determined as $\Gamma_i = \sum_{j=1}^{n} m_j \phi_{j,i}$. In addition, under the two above-mentioned assumptions, it is worth noting that the denominator of Equation 8 is $\boldsymbol{\Gamma}^{\mathrm{T}} \boldsymbol{\Gamma} = m_{tot} = \sum_{j=1}^{n} m_j$, where m_{tot} is the total mass of the structure and m_j is the mass related to the j-th DOF. According to (Zhang and Aktan 1998), the cumulative mass participation factor can be then compared against a selected threshold (for example, a threshold value equal to 90%), to decide if the number of considered modes is sufficient to obtain good estimates of the modal flexibility.

In (Bernagozzi et al. 2017a) an approach has been proposed to predict the amount of truncation effects expected on modal flexibility-based deflections due to generic loads and estimated using a limited number of modes. This approach is based on the definition of a parameter, termed load participation factor (LPF), that quantifies the relative contribution of each mode to the modal flexibility-based deflection. In particular, this load participation factor can be evaluated for each i-th mode as follows

$$\chi_{p,(i)} = \frac{c_{p,i}\,\Gamma_i}{\boldsymbol{c}_{\boldsymbol{p}}^T \boldsymbol{\Gamma}} \tag{9}$$

where, as already mentioned, the term Γ_i is the modal participation factor of i-th mode. The term $c_{p,i}$ is defined as $c_{p,i} = \sum_{k=1}^{n} p_k\, \phi_{k,i}$ and it can be considered as the work done by the external load $\boldsymbol{p}$ for the modal displacements of the i-th mode shape $\boldsymbol{\phi}_i$. In addition, referring to Equation 9, it can be demonstrated, as shown in (Bernagozzi et al. 2017a), that the denominator of the expression of the load participation factor is $\boldsymbol{c}_{\boldsymbol{p}}^T \boldsymbol{\Gamma} = p_{TOT} = \sum_{j=1}^{n} p_j$, where p_{TOT} is the summation of all the components of the load $\boldsymbol{p}$ applied to evaluate the MF-based deflection. Then, starting from the LPFs of the single modes, the cumulative load participation factor can be evaluated for the considered modes

$$\chi_{p,r} = \sum_{i=1}^{r} \chi_{p,(i)} \tag{10}$$

and, finally, an estimate of the amount of the truncation effects present on the overall deflection assembled using the first r modes can be obtained by evaluating the term $|\chi_{p,r} - 1|$. As shown in (Bernagozzi et al. 2017a), the approach based on load participation factor can be considered as an extension and generalization of the approach by (Zhang and Aktan 1998), based on mass participation factor. The approach by (Zhang and Aktan 1998) applied for truncation error analysis on deflections is in fact absolutely suitable for the case of uniform loads and structures with a uniform mass distribution. On the contrary, as shown in (Bernagozzi et al. 2017a), the approach based on load participation factor can be applied in the more general cases of deflections evaluated for generic loads and structures with generic mass distributions.

The approaches based on participation factors (i.e. mass or load participation factors) have been presented in this section of this paper to provide an overview of the existing literature approaches. However, such approaches were not applied in the numerical analyses of this paper, since, as already mentioned, the paper aims at reducing the flexibility truncation effects. The paper is not focused on obtaining a prediction of the flexibility truncation effects, which is, basically, the main achievement that can be obtained using the above-mentioned participation factors.

4 REDUCTION OF THE TRUNCATION EFFECTS ON MF-BASED DEFLECTIONS AND INTERSTORY DRIFTS OF FRAME BUILDINGS WITH MASS IRREGULARITIES

4.1 Mass Proportional Load (MPL): an alternative to the commonly adopted Uniform Load (UL)

Analyses presented in the previous work by the authors (Bernagozzi et al. 2017a) showed that the truncation effects on the displacement components of uniform load deflections tend to increase if structures with increasing amounts of mass irregularities are considered. However, it was also shown that, by adopting alternative strategies to estimate the deflections, the above-mentioned truncation effects can be reduced. An effective strategy, adopted in (Bernagozzi et al. 2017a), is to estimate the modal flexibility-based deflections by applying a load that is proportional to the mass distribution of the structure, defined as Mass Proportional Load (MPL). This strategy was also considered in the present paper to study the flexibility truncation errors that affect the MF-based interstory drifts of building structures with irregular in elevation mass distributions.

For structures characterized by a diagonal mass matrix, such as, for example, the structures considered in this paper – i.e. planar shear-type buildings, the mass proportional load, indicated as $\boldsymbol{p}^{\boldsymbol{m}}$, can be expressed as follows

$$\boldsymbol{p}^{\boldsymbol{m}} = \boldsymbol{M}\,\boldsymbol{a} \tag{11}$$

where $\boldsymbol{M}_{n\times n}$ is the mass matrix of the structure and $\boldsymbol{a}_{n\times 1}$ is a unitary acceleration vector with constant terms a assumed equal to one. Thus, the j-th component of the MPL vector is $p_j^m = a\, m_j$. The mass proportional load is a load that has a special property. In fact, if a mass proportional load is considered for evaluating the deflections, then the approach for truncation error analysis based on the load participation factor proposed in (Bernagozzi et al. 2017a) is equivalent to the approach based on the mass participation factor that was presented in the work by (Zhang and Aktan 1998). This property is immediately evident if one evaluates the load participation factor $\chi_{p,(i)}$ related to the i-th mode (i.e. Equation 9) by considering a mass proportional load. In such case the expression that is obtained is equal to the mass participation factor $\mu_{(i)}$ (i.e. Equation 8).

The mass proportional load can be considered as an alternative to the uniform load, which is commonly adopted in the procedures for damage detection that use modal flexibility-based deflections. However, if one evaluates the deflections due to such two loads, the displacements and interstory drifts that are obtained in the two cases are in general not of the same order of magnitude. This is expected since the mass proportional load is a vector composed by the masses of the structure, while the uniform load, as considered either in the work by (Zhang and Aktan 1998) or in the work by (Koo et al. 2010), is a vector whose components are equal to one. To make these two loads more comparable, a scaled version of the uniform load can be considered, as done in (Bernagozzi et al. 2017a). This modified version of the uniform load is defined as follows

$$\boldsymbol{p^u}=m^*\,\boldsymbol{a} \tag{12}$$

where $m^*=\frac{1}{n}\sum_{j=1}^{n} m_j$ is the average mass of the MDOF structure – i.e. the average story mass for the structures considered in this paper. The j-th component of the scaled version of the UL vector is thus $p_j^u=a\,m^*$.

4.2 Analytical expressions for the determination of the truncation errors on the MF-based interstory drifts of shear-type frame buildings

An analytical expression was derived in the previous work by the authors (Bernagozzi et al. 2017a) for a direct determination of the truncation errors that affect the MF-based interstory drifts of shear-type frame buildings due to a generic load. The truncation error related to the interstory drift at the j-th story can be determined as follows

$$\varepsilon_{d,r}^{j}=\alpha_{d,r}^{j}-1=\frac{\sum_{i=1}^{r} c_{p,i}\, g_i^{j,UP}}{V_j}-1 \tag{13}$$

where $\alpha^j_{d,r}$ is the relative contribution of the first r modes to the drift, the term $g_i^{j,UP}=\sum_{k=j}^{n} m_k\,\phi_{k,i}$ is the portion of the participation factor Γ_i extended only to the degrees-of-freedom (i.e. floor levels) that are above the selected j-th story, and V_j is the story shear at the j-th story of the structure – i.e. $V_j=\sum_{k=j}^{n} p_k$. Without going into the details, discussed in (Bernagozzi et al. 2017a), it is worth noting that it was possible to derive this explicit formula (i.e. Equation 13) mainly by taking advantage of the special topology and the properties related to the flexibility matrix of a shear-type building structure. If one performs the calculations on a numerical model of a shear-type building, Equation 13 provides theoretically the same results of Equation 5. However, while in Equation 5 truncated and non-truncated solutions computed from the modal flexibility matrix are considered, in Equation 13 the truncation errors are directly determined starting from the knowledge of the structural masses, the applied load and the components of the mode shapes related to the first r modes.

Equation 13 is used herein to derive the analytical expressions for the determination of the truncation errors on the MF-based interstory drifts evaluated for the two specific loads considered in Section 4.1 – i.e. the mass proportional load (Equation 11) and the uniform load scaled using the average mass, according to Equation 12.

Let us consider at first the mass proportional load. The term $c_{p,i}$ evaluated for the mass proportional load is

$$c_{p,i}^{MPL}= a\sum_{k=1}^{n} m_k\,\phi_{k,i}=a\,\Gamma_i \tag{14}$$

The story shear at the j-th story of the structure V_j evaluated for the mass proportional load can be expressed as

$$V_j^{MPL}=a\sum_{k=j}^{n} m_k \tag{15}$$

Then, Equations 14 and 15 can be substituted into Equation 13, to derive the analytical expression for the determination of the truncation error on the j-th modal flexibility-based interstory drift evaluated for the mass proportional load. This analytical expression is as follows

$$\varepsilon_{d,r}^{j,\,MPL}=\frac{\sum_{i=1}^{r}\Gamma_i\, g_i^{j,UP}}{\sum_{k=j}^{n} m_k}-1 \tag{16}$$

where the constant term a equal to one is not present, since it cancels out.

Similar operations are now performed by considering the uniform load scaled through the average mass (i.e. Equation 12) as the applied load. By considering this scaled version of the uniform load, the term $c_{p,i}$ becomes

$$c_{p,i}^{UL}= a\,m^*\sum_{k=1}^{n}\phi_{k,i}=a\,m^* s_i \tag{17}$$

where the term s_i expresses the modal contribution of the i-th mode and is the summation of the components of the i-th mode shape (i.e. $s_i=\sum_{k=1}^{n}\phi_{k,i}$). Then, the story shear at the j-th story of the structure V_j can be evaluated when considering the scaled version of the uniform load, i.e.

$$V_j^{UL} = \sum_{k=j}^{n} a\, m^* \ = a\, m^* \ (n+1-j) \qquad (18)$$

where, as already mentioned in Section 2, n is the total number of the stories of the shear-type building. Finally, Equations 17 and 18 can be introduced into Equation 13. The analytical expression for the determination of the truncation error on the j-th interstory drift evaluated for the considered scaled version of the uniform load is thus

$$\varepsilon_{d,r}^{j,\ UL} = \frac{\sum_{i=1}^{r} s_i\, g_i^{j,UP}}{n+1-j} - 1 \qquad (19)$$

It is worth noting that in Equation 19 the average mass m^* is not present, since it is a constant term that cancels out (similarly to the constant term a). This also implies that the truncation errors of Equation 19, which are relative errors on the interstory drifts, are not altered if one performs a uniform scaling (for example, using the average mass m^*) on the applied load, which is, in such case, a uniform load.

5 NUMERICAL ANALYSES

5.1 *Numerical analyses on selected structural configurations of a frame building with mass irregularities*

Numerical analyses were performed on models of a building structure with mass irregularities to evaluate the truncation errors that affect the modal flexibility-based interstory drifts due to the uniform load, and to compare such errors with those obtained by applying a mass proportional load. The considered structure is a 6-story reinforced concrete (RC) frame building, which can be modeled as a planar shear-type building (Figure 1). The frame is constituted by three bays, and four columns with a squared cross-section (dimensions: 0.5×0.5 m) are present at each story. The elastic modulus of the concrete is assumed as $E = 3\times10^{10}$ N/m^2. The beams of the frame are supposed to be infinitely stiff in comparison to the columns, and the structure is characterized by a uniform distribution of the story stiffness (i.e. the stiffness is k_j=2.29×10^5 kN/m for each story j=1 … n with n =6).

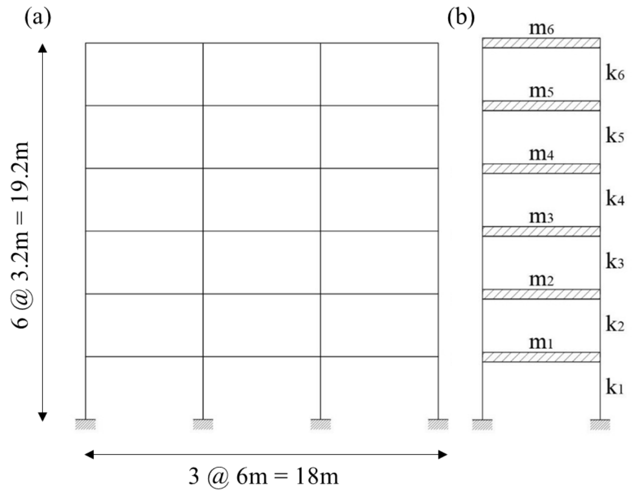

Figure 1. Six-story reinforced concrete planar frame building: (a) elevation; (b) shear-type model of the building - same building model considered in (Bernagozzi et al. 2017a).

Twelve different configurations of the building structure characterized by irregular in elevation distributions of the story masses were considered. Such configurations are reported from no. 1 to no. 12 in Table 1, where the mass distributions are expressed in terms of the story mass ratios. The coefficient γ present in Table 1 can assume the following values: 1, 2, 3, 4, 5, and was used to introduce increasing amounts of mass irregularities on the considered configurations. Each increased mass at the j-th floor level of the building model can be expressed as $m_j = \gamma\ m_{ref}$, where m_{ref} = 100 kN s^2/m. As shown in Table 1, for the different configurations the mass irregularities were imposed in different positions, by considering both single and multiple locations. It is worth noting that the selected building model and the related different configurations are the same that were considered in the previous work by the authors (Bernagozzi et al. 2017a), where, on the contrary, the analyses have been focused on the evaluation of the truncation errors on the displacement components of the MF-based deflections (and not on the interstory drifts). The selected configurations are characterized either by moderate mass irregularities (i.e. more realistic situations) or by strong mass irregularities (which can be considered as more rare configurations). The considered wide variability in the amount of the mass irregularities was, in any case, chosen to have a complete insight of the tendencies of the results. Of course, if a certain configuration of Table 1 has all the γ coefficients equal to one, then the structure has a uniform distribution of the masses, and thus the mass proportional load becomes a uniform load.

Table 1. Distribution of the masses of the considered structural configurations of the building expressed in terms of the story mass ratios, with γ =1,2,3,4,5 - same configurations considered in (Bernagozzi et al. 2017a).

j-th DOF	1	2	3	4	5	6	7	8	9	10	11	12
6	γ	1	1	1	1	1	γ	1	1	1	γ	1
5	1	γ	1	1	1	1	1	γ	1	1	1	γ
4	1	1	γ	1	1	1	γ	1	γ	1	γ	1
3	1	1	1	γ	1	1	1	γ	1	γ	1	γ
2	1	1	1	1	γ	1	1	1	γ	1	γ	1
1	1	1	1	1	1	γ	1	1	1	γ	1	γ

The calculations that were performed in the numerical analyses presented in this section are reported in the flow chart of Figure 2. At first, for each considered configuration (Table 1) an undamped numerical model of the shear-type building structure, expressed in terms of the stiffness and mass matrices, was assembled. The modal parameters – i.e. natural frequencies and mode shapes – were determined through an analytical modal analysis performed on the model. Then, the modal flexibility matrices were assembled, according to Equation 1, and this operation was repeated for all the possible subsets of modes to be included in the calculations (i.e. for $r = 1 \ldots n$, where $n = 6$ for the considered building). The modal flexibility-based deflections were then determined, according to Equation 2, by applying both the mass proportional load and the uniform load (specifically, the uniform load scaled through the average story mass, as defined in Section 4.1). Using Equation 3, the modal flexibility-based interstory drifts were subsequently evaluated starting from the deflections. The interstory drifts evaluated by considering all the modes to assemble the modal flexibility (i.e. for $r = n$) represent the non-truncated solutions (i.e. target solutions) and were used to calculate the errors that affect the truncated interstory drifts evaluated for a number of modes $r = 1 \ldots n - 1$. To obtain the relative truncation errors on the interstory drifts related to each single story, Equation 5 was applied. In addition, the root-mean-square values of the truncation errors on the drifts related to all the stories of the building were also determined, according to Equation 6.

The results of the truncation error analysis are presented in this section using the following strategy. At first, the results are discussed for one structural configuration of the shear-type building (i.e. configuration 7) with a mass irregularity that

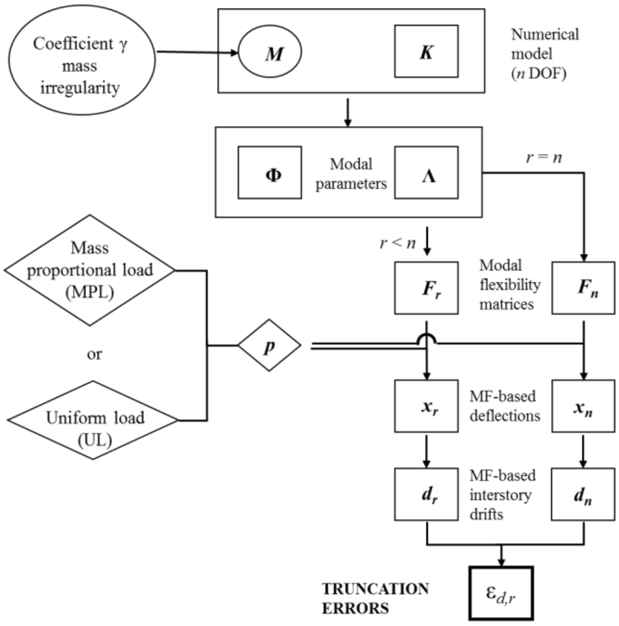

Figure 2. Flow chart of the analysis carried out to compare the mass proportional load and the uniform load by evaluating the corresponding truncation errors.

is imposed by selecting one value of the coefficient γ (i.e. γ=3). Then, by considering the same structural configuration (i.e. configuration 7) the results are presented for increasing amounts of the mass irregularities (i.e. for all the considered values of the coefficient γ, i.e. 1, 2, 3, 4, 5). At the end, the results are shown for all the structural configurations reported in Table 1 (configurations from 1 to 12) and by considering a fixed mass irregularity (again imposed with γ=3).

The modal flexibility-based interstory drifts obtained for configuration 7 of the shear-type building with a mass irregularity imposed using γ=3 are presented in Figure 3. The drifts due to the uniform load are reported on the left-hand side (Figure 3a), while the drifts due to the mass proportional load are on the right-hand side (Figure 3b). In both figures the results are shown for all the possible subsets of modes to be included in the modal flexibility (i.e. for all the different values assumed by the parameter r). Referring to the non-truncated solutions (obtained for $r = n = 6$), the interstory drifts due to the uniform load increase linearly from the upper to the lower stories of the building. This trend is expected since the structure has a uniform distribution of the story stiffness, and the story shear induced by the uniform load increases linearly, as well, from the upper to the lower stories. This trend related to the drifts due to the uniform load is, on the contrary, not observed for the drifts due to the mass proportional load, as expected. Referring to the truncated solutions, in Figure 3a it is evident that the profiles of the drifts due to the uniform load calculated for r=1 (i.e. using only the first mode) or r=2 (i.e. using only the first two modes) show the major discrepancies with respect to the non-truncated drifts. On the contrary, all the profiles of the drifts due to the mass proportional load evaluated using a limited number of modes (for $r < n$) are close to the corresponding non-truncated drifts obtained for $r = n = 6$, as evident in Figure 3b.

Figure 4 shows the percent truncation errors that affect the modal flexibility-based interstory drifts of each story of the structure, which were evaluated with respect to the non-truncated solutions obtained for r=n. These errors are reported in terms of absolute values in Figure 4a and 4b, which are related to the calculations performed using the uniform load and the mass proportional load, respectively. For both the two applied loads, it is shown in Figure 4 that the errors on the interstory drifts at the upper stories of the building are in general higher than the errors related to drifts at the bottom of the structure. This trend is closely related to the fact that, for the considered structural configuration, the drifts at the upper stories are lower than the drifts at the bottom of the building, and thus the drifts at the upper stories are more significantly affected by the modal truncation effects. However, the important result of Figure 4 to be highlighted is that the truncation errors on the drifts due to the mass proportional load are in general lower than the errors on the drifts due to the uniform load. This is evident by considering the profiles of the drifts obtained for r = 1, 2, 3, 4, and especially by considering the drifts at the upper stories of the building.

The percent truncation errors that are shown in Figure 4 by highlighting the trend of such errors along the height of the building are also plotted in Figure 5 but using a different strategy. In Figure 5 errors related to each interstory are plotted separately, and the trend of the truncation errors is presented as a function of the number of the considered modes (i.e. the parameter r). As expected, by considering increasing values of the parameter r, the truncation errors for both the uniform and the mass proportional loads in general get lower. However, it can be observed that the decreasing trends of the errors related to the mass proportional load seem to be more regular than the ones related to the uniform load, as shown for example in Figure 5a for j=6. In addition, as already discussed for Figure 4, the results show that the errors related to the mass proportional load are lower than the errors related to the uniform load for the majority of the interstory drifts and the

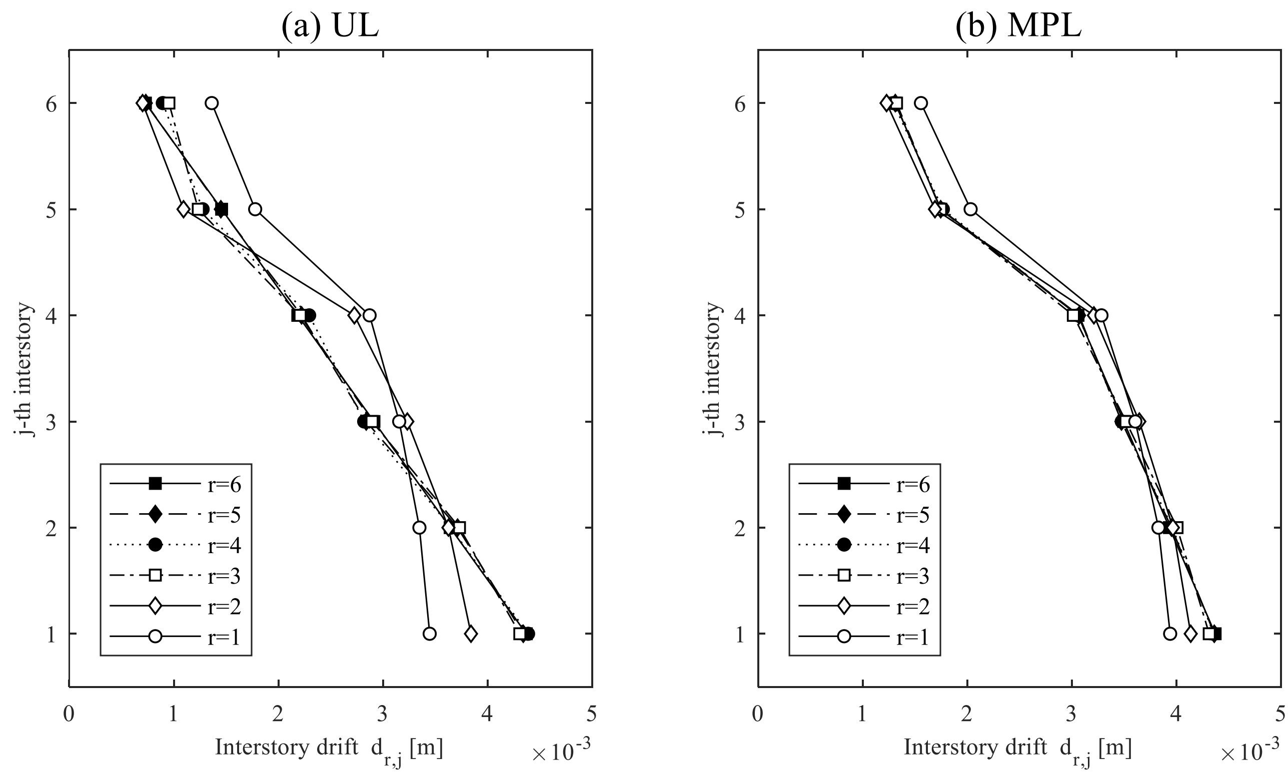

Figure 3. Interstory drifts evaluated from the modal flexibility-based deflections of the building structure (configuration 7 with γ=3): (a) uniform load; (b) mass proportional load.

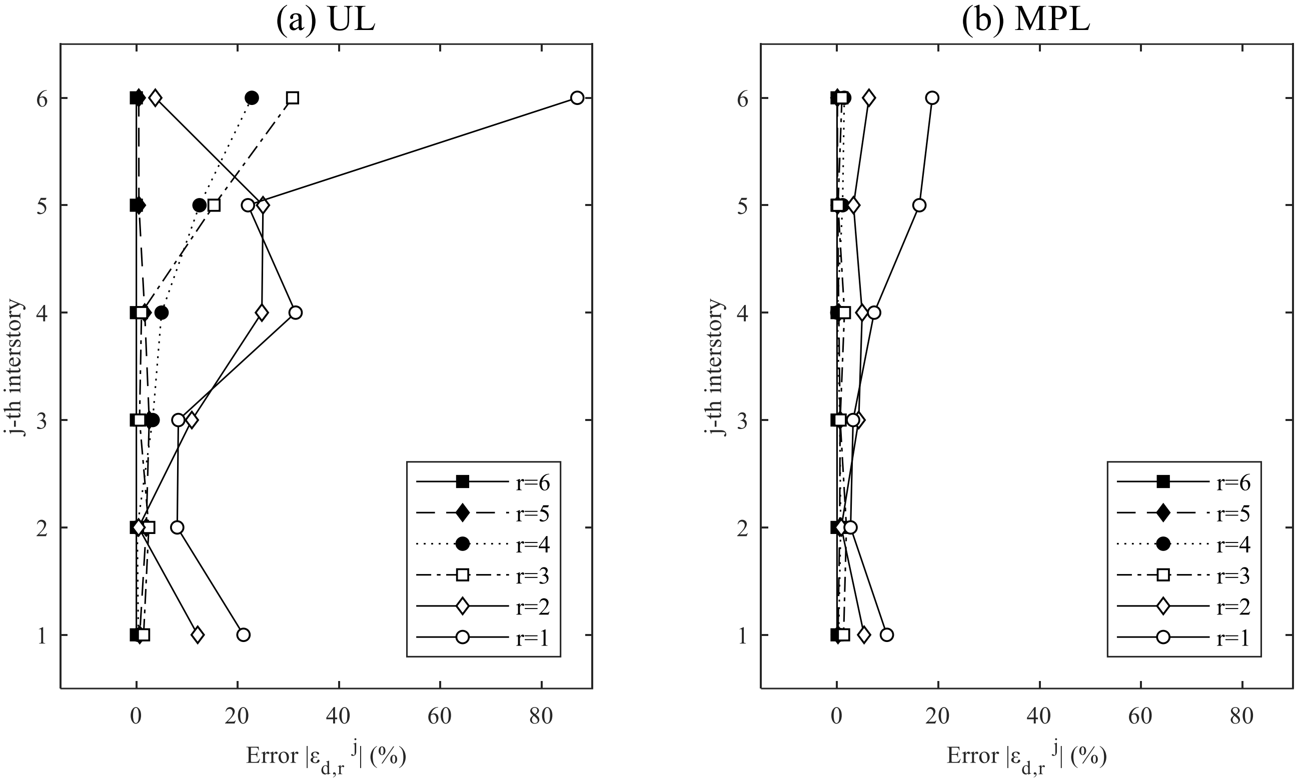

Figure 4. Truncation errors on the interstory drifts of the building structure (configuration 7 with γ=3): (a) uniform load; (b) mass proportional load.

values of the parameter *r*. In the few cases where this does not happen, the errors related to the mass proportional and the uniform loads are comparable, as evident in Figure 5.

The results of the analyses performed by considering again configuration 7 of the shear-type building but for increasing amounts of mass irregularities (i.e. γ =1, 2, 3, 4, 5) are presented in Figure 6, where the truncations effects are quantified by the root-mean-square (RMS) values of the errors related to the single interstory drifts. In Figure 6 the RMS errors on the drifts are plotted as a function of the parameter *r* (reported on the x axis), and each of the different curves is related to one value of the parameter γ. Again, the results obtained using the uniform load (Figure 6a) are compared with the ones obtained using the mass proportional load (Figure 6b). As shown in Figure 6, it is evident that, for all the analyzed cases, the RMS errors on the drifts related to the mass proportional load are lower than (or at least equal to) the errors related to the uniform load. A convenient and immediate way to perceive this result is, for example, to consider the curve of the RMS errors obtained for γ = 1 (which is, of course, equal for the uniform and the mass proportional loads) as a reference curve to compare the two graphs. Using this strategy, it is evident in Figure 6a that all the uniform load RMS errors obtained for γ=2, 3, 4, 5 are located above the curve γ =1. On the contrary, as shown in Figure 6b, all the mass proportional load RMS errors obtained for γ=2, 3, 4, 5 are located below the curve γ =1.

The results obtained for all the structural configurations (from 1 to 12) of the shear-type building by considering a mass irregularity imposed with γ=3 are shown in Table 2, where the RMS truncation errors on the drifts due to the uniform load are compared with respect to the errors related to the mass proportional load. It is worth noting that in Table 2 the column of the results for *r*=6 has been omitted to preserve space, and since, of course, for the considered 6-story shear-type structure for *r*=6 the errors related to both the UL and the MPL are equal to zero. The total number of the analyzed cases is 72, which is calculated as follows: 12 configurations × 6 values of the parameter *r* (including the case *r*=6). For the majority (i.e. 87.5%) of these analyzed cases, as shown in the table, the mass proportional load RMS truncation errors are lower than (or at least equal to) the errors related to the uniform load. Some systematic results can be observed on the RMS truncation errors related to interstory drifts obtained by including only the first structural mode in the calculations (i.e. for *r*=1). For *r*=1 the

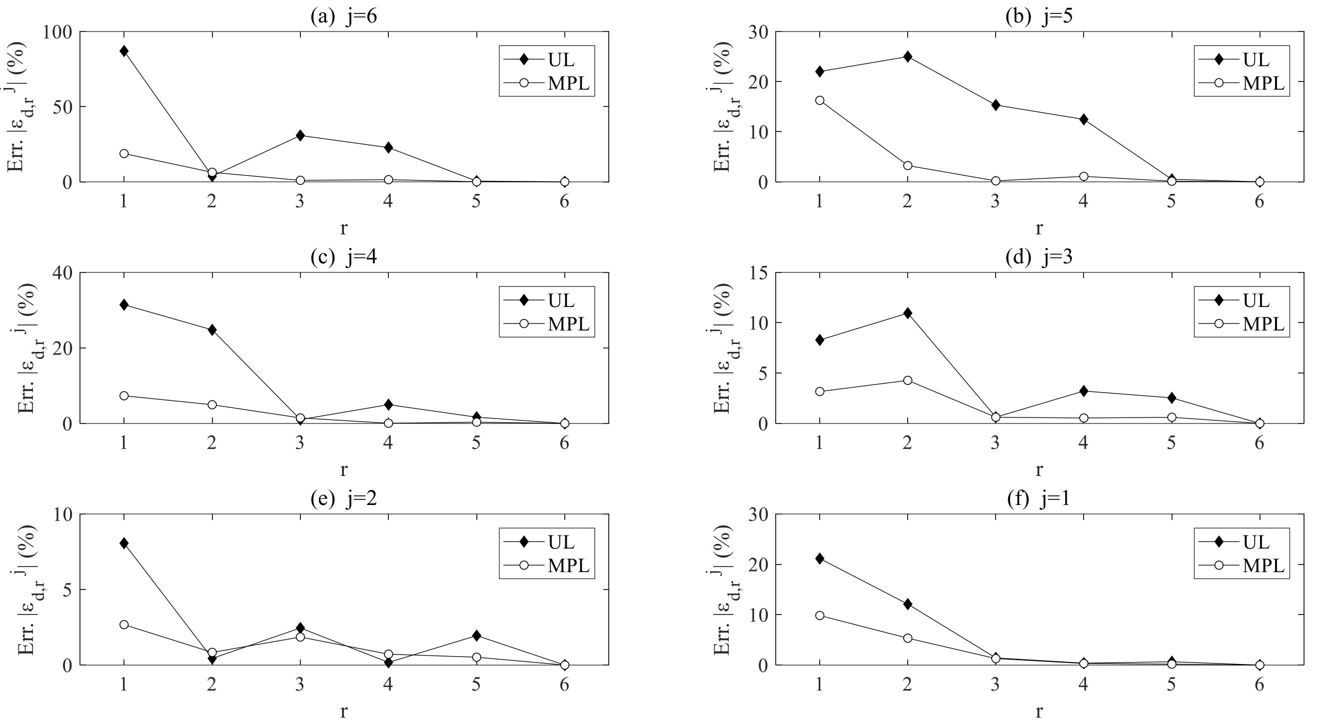

Figure 5. Truncation errors plotted separately for each *j*-th interstory drift of the building structure (configuration 7 with γ=3) – comparison between the UL and the MPL.

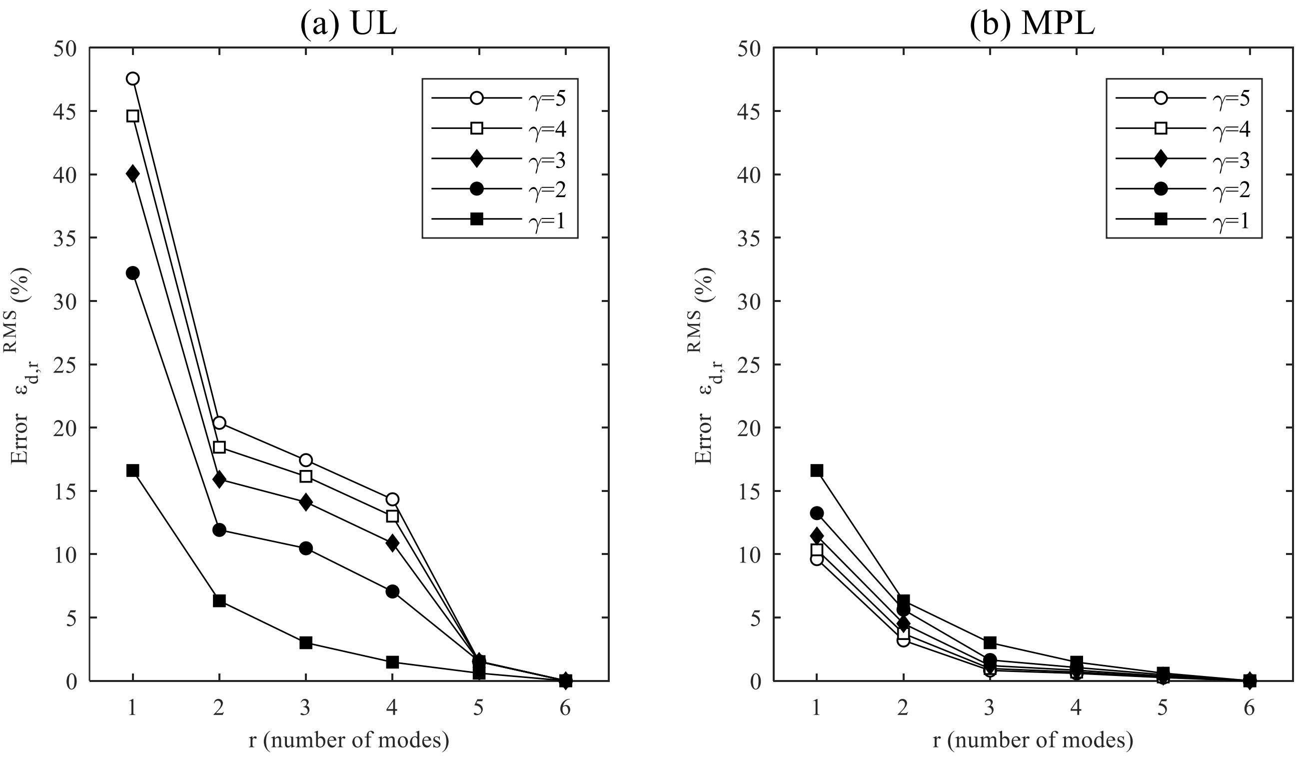

Figure 6. RMS truncation errors on the interstory drifts of the building structure (configuration 7) for different mass irregularities: (a) uniform load; (b) mass proportional load.

results show that in general the MPL errors are lower than the UL errors, if the mass increments are imposed at the upper DOFs of the building (as evident, for example, for configurations 1,2,3,7,8,11). On the contrary, again for r=1, when the mass increments are imposed at the lower DOFs of the building, the UL errors are in general lower than the MPL errors (with only one exception – i.e. configuration 12, for which, even if the mass irregularities are applied at the lower stories, the errors related to the UL and MPL loads are comparable for r=1). It is worth noting that these general trends observed for the analyses conducted by including only the first mode to estimate the MF-based interstory drifts are the same general trends that were observed in the previous work by the authors (Bernagozzi et al. 2017a) in the analyses performed on the displacement components of deflections assembled again with r=1. When considering, on the contrary, the RMS truncation errors obtained for higher values of the parameter r (i.e. for $r > 1$), it is evident in Table 2 that the MPL errors are in general lower than the UL errors (with only very few exceptions, for which, in any case, the percentages of the MPL and UL truncation errors are comparable).

Table 2. RMS truncation errors on the interstory drifts of the building structure – different structural configurations with a fixed mass irregularity ($\gamma = 3$).

Con.	Load	RMS errors on drifts $\varepsilon_{d,r}^{RMS}$ (%)				
		r=1	r=2	r=3	r=4	r=5
1	UL	60.6	6.0	7.7	0.9	1.1
	MPL	12.3	3.7	1.6	0.7	0.3
2	UL	27.8	24.3	7.9	3.6	0.1
	MPL	11.6	4.6	5.3	1.1	0.4
3	UL	17.3	23.5	5.8	5.5	1.5
	MPL	12.6	9.6	1.8	0.9	0.5
4	UL	10.7	9.2	11.2	2.5	2.3
	MPL	16.3	4.9	2.8	0.8	0.6
5	UL	5.7	6.2	4.4	7.1	1.5
	MPL	22.0	3.7	2.2	2.3	0.1
6	UL	12.3	4.3	3.3	1.7	0.7
	MPL	24.7	7.4	1.4	0.4	0.1
7	UL	40.1	15.9	14.1	10.9	1.5
	MPL	11.4	4.5	1.2	0.8	0.4
8	UL	20.7	21.7	9.7	9.5	0.3
	MPL	13.3	3.5	5.9	0.7	0.6
9	UL	16.4	18.2	7.0	6.9	9.4
	MPL	17.1	5.7	1.5	1.5	0.5
10	UL	10.6	8.5	6.7	4.1	2.9
	MPL	22.1	11.7	0.8	0.4	0.1
11	UL	33.8	11.6	13.2	10.4	12.3
	MPL	15.6	4.9	1.1	1.2	0.6
12	UL	19.7	20.6	18.3	8.5	8.0
	MPL	17.8	7.3	0.9	0.3	0.2

5.2 Parametric studies on frame buildings with a different number of stories

In this section the results of parametric studies for truncation error analysis performed on frame buildings with a different number of stories are presented. The considered structures are shown in Figure 7, and such structures are, respectively, a 2-story frame, a 3-story frame, a 4-story frame, a 5-story frame, and a 6-story frame. For each structure the following parametric study is performed. It is assumed that each floor level can be characterized by a value of the story mass ratio expressed by the coefficient γ, and that such coefficient can take each of the following discrete values: 1, 2, 3, 4, 5. Under these assumptions, for each structure all the possible combinations of mass distributions (i.e. all the possible combinations of the story mass ratios at the different floor levels) are analyzed. In general, for a n story structure the number of the considered combinations of mass distributions is equal to N^n, where N is the number of the different values that the coefficient γ can assume (i.e. equal to 5 in such case).

By considering all these different combinations of mass distributions, the purpose is to quantify the cases (i.e. the percentages of such cases evaluated with respect to the total number of analyzed combinations) for which the mass proportional load provides truncation errors on the modal flexibility-based interstory drifts that are lower

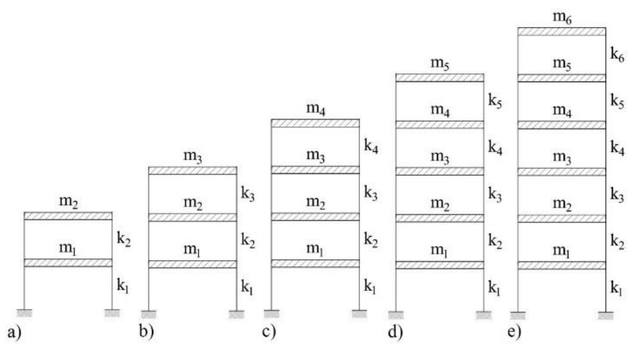

Figure 7. Shear-type structural models with a different number of stories considered in the parametric study: a) 2-story frame; b) 3-story frame; c) 4-story frame; d) 5-story frame; e) 6-story frame.

than (or at least equal to) the errors related to the application of the uniform load. If the comparison is made between truncation errors evaluated for a single interstory of the building, the percentage of cases for which $|\varepsilon_r^{j,MPL}| \leq |\varepsilon_r^{j,UL}|$ is indicated with the symbol η_r^j, where r is the number of the included modes. It is worth noting that in the above-mentioned expression the subscript d, which according to the notation used in the paper stands for drifts, has been omitted for simplicity. If the comparison is made between errors averaged along the height of the building using the root-mean-square criterion, the percentage of cases for which $\varepsilon_r^{RMS,\ MPL} \leq \varepsilon_r^{RMS,\ UL}$ is indicated as η_r^{RMS}. For the purposes of the present study, the percentages η_r^j and η_r^{RMS} thus denote positive cases, for which the MPL provides better (or at least equal) performance with respect to the UL.

In the performed parametric study several configurations of the considered building models were taken into account, and to calculate the errors on the drifts the analytical expressions derived in Section 4.2 have been used in place of Equation 5. This was a convenient choice that has been made to reduce the computational costs of the analyses. In fact, as already mentioned in Section 4.2, the derived analytical expressions for evaluating the truncation errors on the MF-based interstory drifts of shear-type buildings have the advantage that there is no need to assemble the modal flexibility matrices in the calculations. In particular, in the performed parametric study the analytical expressions specifically derived for the mass proportional load (i.e. Equation 16) and the uniform load (i.e. Equation 19) have been adopted.

The results of the parametric study are presented at first by comparing the truncation errors related to single interstory drifts and by considering the case of the 6-story structure. Then, the results are presented by comparing the truncation errors averaged using the RMS criterion and by presenting the results for all the considered buildings with a different number of stories.

Table 3 shows the percentages of cases for which $|\varepsilon_r^{j,MPL}| \leq |\varepsilon_r^{j,UL}|$ for analyses performed on the single interstory drifts of the 6-story structure. The number of the considered combinations of mass distributions for the 6-story building is equal to 15625 (i.e. N^n, with N=5 and n=6). The percentages η_r^j are presented in Table 3 for different values of j (from 1 to 6) and for different values of the parameter r (from 1 to 5). Of course, for r=6 there are no truncation errors, and thus the performance of MPL is ideally equal to the UL, with values of η_r^j for r=6 that are 100% for each interstory j. In Table 3 the column of the results η_r^j for r=6 is thus not shown, to preserve space. As

Table 3. Percentages of cases (%) for which $|\varepsilon_r^{j,MPL}| \leq |\varepsilon_r^{j,UL}|$: 6-story frame – 15625 combinations.

j-th story	η_1^j	η_2^j	η_3^j	η_4^j	η_5^j
6	71.4	78.1	86.9	92.7	92.3
5	55.6	76.4	91.4	93.9	92.9
4	49.1	88.3	84.5	93.1	92.8
3	45.8	74.0	87.7	88.3	92.6
2	66.6	53.6	76.9	89.3	92.2
1	53.4	62.8	75.4	89.1	92.2

evident in Table 3, the percentages of positive cases, for which the MPL provides better (or at least equal) performance with respect to the UL, are in general high. Such percentages tend to increase as the number of included modes r increases, and only few of these percentages are approximately around 50% especially for r=1.

Figure 8 and Table 4 show the percentages of cases for which $\varepsilon_r^{RMS,\ MPL} \leq \varepsilon_r^{RMS,\ UL}$, where truncation errors averaged along the height of the buildings using the RMS criterion are considered. In particular, Figure 8 shows the results obtained for the 6-story frame in a stacked bar plot, where the dark grey bars represent the percentages η_r^{RMS} (i.e. positive cases). While, Table 4 shows the results obtained for all the analyzed buildings with a different number of stories, and the percentages η_r^{RMS} are reported as a function of the parameter r, which, for each different structure with n stories,

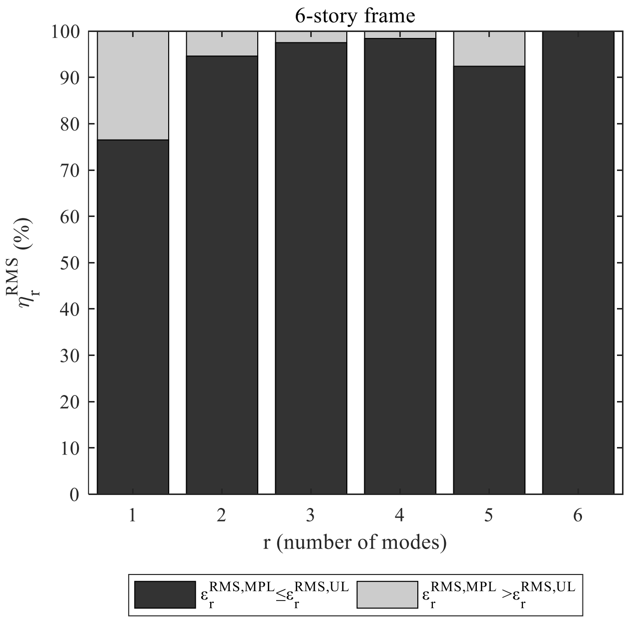

Figure 8. Analysis of the RMS truncation errors on the interstory drifts of the 6-story frame; evaluation of the percentages for which $\varepsilon_r^{RMS,\ MPL} \leq \varepsilon_r^{RMS,\ UL}$ (15625 combinations).

Table 4. Percentages of cases (%) for which $\varepsilon_r^{RMS, MPL} \leq \varepsilon_r^{RMS, UL}$: results for all the analyzed buildings with a different number of stories.

n	#comb.	η_1^{RMS}	η_2^{RMS}	η_3^{RMS}	η_4^{RMS}	η_5^{RMS}
2	25	76.0	100.0	-	-	-
3	125	76.8	87.2	100.0	-	-
4	625	74.7	92.6	87.7	100.0	-
5	3125	76.0	94.1	96.0	90.8	100.0
6	15625	76.5	94.6	97.5	98.4	92.4

varies from 1 to n. It is worth noting that in Table 4 the column of the results η_r^{RMS} for r=6 has been omitted to preserve space, and since, of course, for the 6-story structure the percentage η_r^{RMS} for r=6 is equal to 100%. As evident in Figure 8 and Table 4, the percentages of positive cases, for which the mass proportional load provides better (or at least equal) performance with respect to the uniform load, are in general very high, especially when at least the first two structural modes of the considered buildings are included in the calculations (i.e. for $r \geq 2$).

6 CONCLUSIONS

The work has been dedicated to the study of the modal truncation errors that affect the interstory drifts computed from modal flexibility-based deflections of shear-type buildings with irregular in elevation mass distributions. In particular, an attempt was made to reduce the modal truncation errors specifically on the interstory drifts since such parameters are considered as damage-sensitive features in some damage detection procedures developed for building structures, as shown, for example, in (Koo et al. 2010). According to these procedures, the deflections and related drifts are usually computed from the modal flexibility by applying uniform inspection loads. On the contrary, in the analyses of the present paper the interstory drifts have been computed by applying an alternative load, which is defined as mass proportional load. This load has been proposed in a previous work by the authors, i.e. (Bernagozzi et al. 2017a), where, on the contrary, some analyses have concerned the evaluation and reduction of the modal truncation effects on the displacement components of the deflections.

In the present paper, numerical analyses have been conducted on shear-type models of mass irregular building structures. In a first analysis twelve configurations with different mass distributions of a 6-story frame and, specifically, with increasing amounts of the mass irregularities have been considered. In a second analysis five building structures with a different number of stories (from two to six stories) have been selected, and then analyzed by considering all the different combinations of mass distributions that derive from selecting a range of potential values for the story mass ratios. For all the performed analyses, the truncation errors on interstory drifts evaluated for the mass proportional load have been compared with those derived from the uniform load. In general, the results of all the performed analyses showed that for the vast majority of the considered structural configurations, the errors related to the mass proportional load are lower than the errors of the uniform load. In the few cases where this does not happen, the errors are at least comparable with each other.

Based on the obtained results, it seems that the mass proportional load can be considered as a good alternative to the uniform load, when considering buildings with irregular in elevation mass distributions and for the specific purpose of reducing the truncation effects on the modal flexibility-based interstory drifts. It is expected in fact that reducing the errors on these parameters, commonly assumed as damage-sensitive features, might also be beneficial for reducing the impact that the modal truncation effects might have on other quantities related to the damage detection process, such as, for example, damage indices and metrics. As a final remark, it is worth noting that the mass proportional load is a load that, intrinsically, is system dependent. It is thus expected that the mass proportional load might have an applicability, as inspection load in the damage detection procedures based on modal flexibility-based deflections and interstory drifts, especially for building structures for which the masses are unchanged before and after the occurrence of potential damage.

REFERENCES

Aenlle, M.L., Fernández, P., Brincker, R., Fernández-Canteli, A., 2010. Scaling-factor estimation using an optimized mass-change strategy, *Mechanical Systems and Signal Processing*, **24**(5), 1260-1273.

Bernagozzi, G., Landi, L., Diotallevi, P.P., 2017a. Truncation error analysis on modal flexibility-based deflections: application to mass regular and irregular structures, *Engineering Structures*, **142**, 192–210.

Bernagozzi, G., Ventura, C.E., Allahdadian, S., Kaya, Y., Landi, L., Diotallevi, P.P., 2017b. Application of modal flexibility-based deflections for damage diagnosis of a steel frame structure, in: Vestroni, F., Gattulli, V., Romeo, F., (editors), *Procedia Engineering, X*

International Conference on Structural Dynamics EURODYN 2017, volume 199, pp. 2026-2033.

Bernagozzi, G., Mukhopadhyay, S., Betti, R., Landi, L., Diotallevi, P.P., 2018. Output-only damage detection in buildings using proportional modal flexibility-based deflections in unknown mass scenarios, *Engineering Structures*, **167**, 549-566.

Bernal, D., 2004. Modal scaling from known mass perturbations, *Journal of Engineering Mechanics*, **130**(9), 1083 -1088.

Brincker, R., Ventura, C., 2015. *Introduction to OPERATIONAL MODAL ANALYSIS*, 1st ed. Chichester, UK, John Wiley & Sons.

Farrar, R., Worden, K., 2013. *Structural health monitoring: A machine learning perspective*, 1st ed. Chichester, UK, John Wiley & Sons.

Koo, K.Y., Lee, J.J., Yun, C.B., Kim, J.T., 2008. Damage detection in beam-like structures using deflections obtained by modal flexibility matrices, *Smart Structures and Systems*, **4**(5), 605-628.

Koo, K.Y., Sung, S.H., Park, J.W., Jung, H.J., 2010. Damage detection of shear buildings using deflections obtained by modal flexibility, *Smart materials and structures*, **19**(11), 115026.

Koo, K.Y., Sung, S.H., Jung, H.J., 2011. Damage quantification of shear buildings using deflections obtained by modal flexibility, *Smart materials and structures*, **20**(4), 045010.

Sohn, H., Farrar, C.R., Hemez, F.M., Shunk, D.D., Stinemates, D.W., Nadler, B.R., Czarnecki, J.J., 2003. *A Review of Structural Health Monitoring Literature: 1996-2001*, Report LA-13976-MS, Los Alamos National Laboratory, Los Alamos, NM, USA.

Zhang, Z., Aktan, A.E., 1998. Application of Modal Flexibility and Its Derivatives in Structural Identification. *Research in Nondestructive Evaluation*, **10**(1), 43-61.